MTH 264 Introduction to Matrix Algebra - Summer 2023.

LN4A. The Eigenvalue Problem.

These lecture notes are mostly lifted from the text **Matrix and Power Series, Lee and Scarborough, custom 5th edition**. This document highlights parts of the text that are used in the lecture sessions.

Definition 4A.1. Eigenvalues and Eigenvectors

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. A non-zero vector $\mathbf{v} \in \mathbb{R}^n$ is a **eigenvector** of $\mathbf{A}$ if and only if $\mathbf{Av} = \lambda\mathbf{v}$ for some scalar $\lambda \in \mathbb{R}$. The corresponding λ is called the **eigenvalue** corresponding to the eigenvector $\mathbf{v}$. Conversely, we may say that $\mathbf{v}$ is an eigenvector corresponding to λ. The **eigenspace** corresponding to the eigenvalue λ, typically denoted E_{λ_0}, is given by

$$E_{\lambda_0} = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{Av} = \lambda_0\mathbf{v}\}$$

i.e. the collection of all eigenvectors corresponding to λ_0 including the zero vector. The problem of finding all eigenvalues and corresponding eigenvectors, or equivalently, a linearly independent spanning set of the corresponding eigenspace, of $\mathbf{A}$ is called the **eigenvalue problem**.

Definition + Theorem 4A.2. Characteristic Equation determines Eigenvalues

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. The **characteristic polynomial** of $\mathbf{A}$ is the polynomial in λ given by $\det(\mathbf{A} - \lambda\mathbf{I_n})$. The **characteristic equation** of $\mathbf{A}$ is the equation $\det(\mathbf{A} - \lambda\mathbf{I_n}) = 0$. Then, λ_0 is an eigenvalue of $\mathbf{A}$ if and only if λ_0 is root of the characteristic polynomial, i.e. λ_0 satisfies $\det(\mathbf{A} - \lambda\mathbf{I_n}) = 0$.

The **algebraic multiplicity** of λ_0 is its multiplicity as a root of the characteristic polynomial. The **geometric multiplicity** is the maximum cardinality of a linearly independent set of vectors in the eigenspace E_{λ_0} corresponding to λ_0. Note that the algebraic multiplicity of λ_0 is $\geq$ the geometric multiplicity of λ_0.

Observe that based on the characteristic polynomial definition, eigenvalues may be real numbers or complex numbers. For this course, we will only focus on real eigenvalues. To find the real eigenvalues, recall the following result from algebra.

Theorem 4A.3. Synthetic Division

Let $p(x)$ be a polynomial of degree n with real number coefficients. Consider $x - b$ for some $b \in \mathbb{R}$. Then, there exists a polynomial $q(x)$ of degree $(n-1)$ and a scalar $r \in \mathbb{R}$ such that

$$\frac{p(x)}{x-b} = q(x) + \frac{r}{x-b}$$

Then, $r = 0$ if and only if $x = b$ is a root of $p(x)$. **Polynomial long division** can be used to find $q(x)$ and r.

We can simulate polynomial long division of $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x_1 + a_0$ with $a_n \neq 0$ by $(x - b)$ using **synthetic division**, given below:

$$\begin{array}{c|cccccc} b & a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ & & b\cdot q_{n-1} & b\cdot q_{n-2} & \cdots & b\cdot q_1 & b\cdot q_0 \\ \hline & q_{n-1} & q_{n-2} & q_{n-3} & \cdots & q_0 & r \end{array} \quad + \quad \text{with} \quad \begin{aligned} q_{n-1} &= a_n \\ q_{i-1} &= a_i + b\cdot q_i \text{ with } i = n-2, \cdots, 0 \\ r &= a_0 + b\cdot q_0 \end{aligned}$$

Then, $q(x)$ is given by $q(x) = q_{n-1}x^{n-1} + q_{n-2}x^{n-2} + \cdots + q_1 x + q_0$ and r is given as above.

For a more detailed explanation, see Wikipedia: Synthetic Division.

If $p(x)$ has integer coefficients, then possible roots can be found using the **Rational Root Theorem**. If $p(x)$ has a rational root $x = \frac{k}{l}$ for relatively prime $k, l \in \mathbb{Z}$, then k is an integer factor of the constant term a_0 and l is an integer factor of the leading coefficient a_n.

Then, to finding eigenvectors and eigenspaces:

Theorem 4A.4. Finding Eigenvectors

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. The eigenspace E_{λ_0} corresponding to a real eigenvalue λ_0 is exactly the solution set of the system $(\mathbf{A} - \lambda_0 \mathbf{I_n})\mathbf{x} = \mathbf{0}$. Without loss of generality, assume $\{x_1, \ldots, x_k\}$ is a set of free variables of the system. Then, expressing the solution set as $\{x_1\mathbf{v_1} + \cdots + x_k\mathbf{v_k} : x_1, \ldots, x_k \in \mathbb{R}\}$ for some $\mathbf{v_1}, \ldots, \mathbf{v_k} \in \mathbb{R}^n$ yields a linearly independent spanning set of eigenvectors $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ of E_{λ_0}.

Theorem 4A.5. Eigenvectors related to Distinct Eigenvalues

Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of $\mathbf{A} \in \mathbb{R}^{n\times n}$. Let $V = \{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ be a set in $\mathbb{R}^n$ such that $\mathbf{v_i}$ is an eigenvector corresponding to the eigenvalue λ_i. Then, V is linearly independent.

That is, eigenvectors corresponding to distinct eigenvalues form a linearly independent set.

Here are also some properties of eigenvalues and eigenvectors that may make our calculations easier.

Theorem 4A.6. Properties involving the Eigenvalue Problem

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a square matrix.

- **(a)** Let $k \in \mathbb{R}$ be nonzero. Then, λ_0 is an eigenvalue of $\mathbf{A}$ if and only if $k\lambda_0$ is an eigenvalue of $k\mathbf{A}$. Furthermore, the eigenspace E_{λ_0} of $\mathbf{A}$ is exactly the eigenspace $E_{k\lambda_0}$ of $k\mathbf{A}$, i.e. their respective eigenspaces match.
- **(b)** Assume $\mathbf{A}$ is triangular. Then, the diagonal elements of $\mathbf{A}$ are the eigenvalues of $\mathbf{A}$.
- **(c)** $\mathbf{A}$ is invertible if and only if $\mathbf{A}$ does not an eigenvalue of 0.
- **(d)** Assume $\mathbf{A}$ is invertible. Then, λ_0 is an eigenvalue of $\mathbf{A}$ if and only if λ_0^{-1} is an eigenvalue of $\mathbf{A}^{-1}$. Furthermore, their respective eigenspaces match.
- **(e)** Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. Then, λ_0 is an eigenvalue of $\mathbf{A}$ if and only if λ_0 is an eigenvalue of $\mathbf{A}^\top$. However, their respective eigenspaces generally do not match.
- **(f)** Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and let $k \in \mathbb{R}$ be any scalar. Then, λ_0 is an eigenvalue of $\mathbf{A}$ if and only if $\lambda_0 - k$ is an eigenvalue of $\mathbf{A} - k\mathbf{I_n}$. Furthermore, their respective eigenspaces match.
- **(g)** Let $k \in \mathbb{Z}$ with $n \geq 0$. If λ_0 is an eigenvalue of $\mathbf{A}$, then λ_0^k is an eigenvalue of $\mathbf{A}^k$.

One reason to care about eigenvectors is this:

Definition 4A.7. Diagonalizable Matrices

A $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a **diagonal** matrix if all entries not on the main diagonal are zeros. That is, a matrix $\mathbf{A}$ is diagonal if and only if it is both upper triangular and lower triangular.

A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is **diagonalizable** if and only if there exists an invertible matrix $\mathbf{B} \in \mathbb{R}^{n \times n}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{B}^{-1}\mathbf{A}\mathbf{B} = \mathbf{D}$.

Equivalently, a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is **diagonalizable** if and only if there exists a basis $V = \{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ such that each $\mathbf{v_i}$ is an eigenvalue of $\mathbf{A}$. That is, the following equation is true:

$$\begin{pmatrix} | & | & & | \\ \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_n} \\ | & | & & | \end{pmatrix}^{-1} \mathbf{A} \begin{pmatrix} | & | & & | \\ \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_n} \\ | & | & & | \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

with λ_i the eigenvalue corresponding to the eigenvector $\mathbf{v_i}$.

Then, we discuss a special family of matrices called symmetric matrices.

Definition 4A.8. Symmetric Matrices

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. Then, $\mathbf{A}$ is **symmetric** if and only if $\mathbf{A} = \mathbf{A}^\top$.

Theorem 4A.9. The Eigenvalue Problem on Symmetric Matrices

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then,

(a) All eigenvalues of $\mathbf{A}$ are real.

(b) Eigenvectors corresponding to distinct eigenvalues are mutually orthogonal.

(c) There exists a set of n linearly independent set of eigenvectors. Furthermore, that set is pairwise orthogonal. That is, the matrix $\mathbf{A}$ is diagonalizable.

Remark: The text uses the term “mutually orthogonal” for “pairwise orthogonal”.

Theorem 4A.10. Reflections and Orthogonal Projections on $\mathbb{R}^3$

Let $\mathbf{A} \in \mathbb{R}^{3 \times 3}$ be a left multiplication matrix for a reflection or orthogonal projection relative to a plane P passing through the origin. Then, $\mathbf{A}$ is a symmetric matrix.